Scattering Coefficients for Wall Impedance Changes in Waveguides*

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Summary—**The Wiener-Hopf technique is used to obtain an exact solution to a two-dimensional scattering problem. In the problem solved, an incident TE_{10} mode, traveling from $z=-\infty$ in the positive z direction, is confined by infinite bounding planes; these planes have infinite conductivity for $z<0$ and an impedance Z_1, for $z>0$. The scattering from the junction at $z=0$ gives rise to reflection and transmission coefficients that are exactly determined. An approximate solution for the reflection coefficients is also given when the TE_{10} mode is incident from the opposite direction. Finally, a table is presented which lists some transmission and reflection coefficients for rectangular and circular waveguides with discontinuities in the wall impedances.**

Introduction

The physical situation considered is illustrated in Fig. 1. An incident TE_{10} mode, traveling in the positive z direction, is confined between infinite parallel planes at $x=\pm a$; the planes have infinite conductivity for $z<0$ and an impedance Z_1 for $z>0$. When the impedance of the confining planes is specified as Z_1, the ratio of the tangential electric vector to the tangential magnetic vector at the surface is Z_1. These vectors are assumed to be orthogonal, and Poynting's vector points from the center into the confining planes. The amplitudes of the dominant modes scattered from the junction $z=0$ are to be determined.

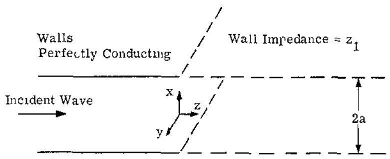

Fig. 1—TE_{10} mode incident in parallel-plane waveguide.

It is well known that the Wiener-Hopf method leads to the solution of the reflection and transmission coefficients for the problem considered here. This problem is quite similar to the duct problem mentioned by Noble[1] and solved by Morse and Feshbach.[2] The method of solution employed here, however, closely follows that used by Papadopoulos for a slightly different geometry.[3]

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[1] B. Noble, "Methods Based on the Weiner-Hopf Technique," Pergamon Press, New York, N. Y., example 3.13, p. 133, and example 3.14, p. 134; 1958.

[2] P. M. Morse and H. Feshbach, "Methods of Theoretical Physics," McGraw-Hill Book Co., Inc., New York, N. Y., p. 1522; 1953.

[3] V. M. Papadopoulos, "Scattering by a semi-infinite strip of dominant mode propagation in an infinite rectangular waveguide," *Proc. Camb. Phil. Soc.*, vol. 52, pp. 553–563; July, 1956.

Method of Solution

The first step in the solution is to determine the eigenvalues and propagation constants on the two sides of the junction. Since only TE_{n0} modes are excited at the junction, we need find these constants only for TE_{n0} modes. When $z<0$, the eigenvalue α_n of the TE_{n0} mode is

$$\alpha_n = (n - 1/2)\pi/a, \qquad z < 0, \tag{1}$$

and the propagation constant C_n is

$$C_n = \sqrt{\alpha_n^2 - k_0^2}, \qquad k_0 = \omega/c, \qquad z < 0. \tag{2}$$

In the region $z>0$, the longitudinal magnetic field of TE_{n0} mode is related to the transverse electric field, E_{yn}, by one of Maxwell's equations

$$H_{zn} = \frac{j}{\omega\mu}\frac{\partial E_{yn}}{\partial x}. \tag{3}$$

Since E_{yn} is cosinusoidal,

$$E_{yn} = \cos \beta_n x e^{\pm P_n z}, \tag{4}$$

the equation satisfied by the eigenvalue β_n is

$$\cot \beta_n a = \frac{-j\beta_n Z_1}{\omega\mu}, \qquad z > 0 \tag{5}$$

because the ratio of the tangential electric to the tangential magnetic vector at the wall is Z_1. The propagation constants P_n for $z>0$ are

$$P_n = \sqrt{\beta_n^2 - k_0^2}, \qquad z > 0. \tag{6}$$

In (4) the plus and minus signs in the exponential term correspond to propagation in the negative and positive z direction respectively.

To apply the Wiener-Hopf technique assume that the transverse electric field is a function of x and z.

$$E_y = F(x, z). \tag{7}$$

Let $F(x, s)$ represent the two-sided Laplace transform of the transverse electric field

$$F(x, s) = \int_{-\infty}^{\infty} F(x, z)e^{-sz}dz. \tag{8}$$

It is now convenient to make the following definitions:

$$F_+ = \int_0^{\infty} F(a, z)e^{-sz}dz, \qquad F_+' = \frac{\partial}{\partial x} F_+ \tag{9}$$

$$F_- = \int_{-\infty}^{0} F(a, z)e^{-sz}dz, \qquad F_-' = \frac{\partial}{\partial x} F_-. \tag{10}$$

Since $F(x, z)$ satisfies the wave equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k_0^2\right) F(x, z) = 0, \tag{11}$$

the application of the two-sided Laplace transform yields

$$\left(\frac{\partial^2}{\partial x^2} + s^2 + k_0^2\right) F(x, s) = 0. \tag{12}$$

Because the transverse electric field is an even function of x, we assume a solution for $F(x, s)$ in the form

$$F(x, s) = C(s) \cos Wx, \tag{13}$$

where s is a complex variable and

$$W = \sqrt{s^2 + k_0^2}.$$

To find the transverse electric field we solve for $C(s)$ using the Wiener-Hopf method, a process involving analytic continuation and Liouville's theorem. The application of the inverse transform then yields E_y.

From (13) we obtain

$$F_+ + F_- = C(s) \cos Wa \tag{14}$$

and

$$F_+' + F_-' = - WC(s) \sin Wa. \tag{15}$$

Because the walls have infinite conductivity when z is negative, the tangential electric field is zero at $x = \pm a$, $z < 0$. Thus,

$$F_- = 0. \tag{16}$$

When $z > 0$ and $x = a$,

$$E_y = Z_1 H_z = \frac{jZ_1}{\omega\mu} \frac{\partial E_y}{\partial x}, \tag{17}$$

hence

$$F_+ = \frac{jZ_1}{\omega\mu} F_+'. \tag{18}$$

From (14)–(16) and (18) we find

$$C(s) = \frac{F_-'}{\dfrac{j\omega\mu}{Z_1} \cos Wa - W \sin Wa}. \tag{19}$$

When this value of $C(s)$ is substituted in (15), the following equation is obtained relating F_+' and F_-'

$$\frac{F_+'}{\dfrac{j\omega\mu}{Z_1} \cos Wa} + \frac{F_-'}{\dfrac{j\omega\mu \cos Wa}{Z_1} - W \sin Wa} = 0. \tag{20}$$

The next step in the Wiener-Hopf procedure is the division of (20) into two parts, each of which is analytic in a half plane. To accomplish this task the denominators of (20) are expanded in infinite products containing the propagation constants.[4]

$$\frac{j\omega\mu}{Z_1} \cos Wa = \frac{j\omega\mu}{Z_1} \cos k_0 a \prod_{n=1}^{\infty} (1 + s/C_n)(1 - s/C_n). \tag{21}$$

$$\frac{j\omega\mu}{Z_1} \cos Wa - W \sin Wa = \left(\frac{j\omega\mu}{Z_1} \cos k_0 a - k_0 \sin k_0 a\right) \cdot \prod_{n=1}^{\infty} (1 + s/P_n)(1 - s/P_n). \tag{22}$$

When these infinite products are substituted in (20) and the result is multiplied by

$$(1 + s/C_1) \prod_{n=1}^{\infty} (1 + s/P_n)(1 - s/C_n),$$

we obtain

$$\frac{F_+'(1 + s/P_1)}{\dfrac{j\omega\mu}{Z_1} \cos k_0 a} \cdot \prod_{n=2}^{\infty} \frac{(1 + s/P_n)}{(1 + s/C_n)} = \frac{-F_-'(1 + s/C_1)}{\left(\dfrac{j\omega\mu}{Z_1} \cos k_0 a - k_0 \sin k_0 a\right)} \cdot \prod_{n=1}^{\infty} \frac{(1 - s/C_n)}{(1 - s/P_n)} \cdot \tag{23}$$

The quantity F_+ on the left has poles at $s = -P_n$ corresponding to the modes propagating toward the right away from the junction; these poles are canceled by the zeros in the infinite product. Thus, the left side of (23) has poles only at $s = -C_n$, $n = 2, 3, 4, \cdots$ and is regular for $\Re(s) > -\Re(C_2)$; ($\Re$ = real part of). Similarly, the right side of (23) is regular for $\Re(s) < \Re(P_1)$, assuming $\Re(P_1) > 0$.

Since there is a common strip of regularity, $-\Re(C_2) < \Re(s) < \Re(P_1)$, the left side of (22) is the analytic continuation of the right side, and both sides are equal to a polynomial in s, according to Liouville's theorem. Due to the left side of (23) being bounded as $s \to +\infty$, the polynomial contains only a constant term B. Hence

$$F_+' = B \frac{j\omega\mu \cos k_0 a}{Z_1(1 + s/P_1)} \prod_{n=2}^{\infty} \frac{(1 + s/C_n)}{(1 + s/P_n)} \cdot \tag{24}$$

[4] R. E. Collin, "Field Theory of Guided Waves," McGraw-Hill Book Co., Inc., New York, N. Y., p. 575; 1960.

Next we obtain

$$F(x, s) = -\frac{B \cos Wx}{(1 + s/C_1)\prod_{n=1}^{\infty}(1 + s/P_n)(1 - s/C_n)} \tag{25}$$

from (24), (14), (16), (18), and (13).

The transverse electric field E_y is obtained by taking the inverse transform of (25)

$$E_y(x, z) = \frac{1}{2\pi j}\int_{s_0 - j\infty}^{s_0 + j\infty} F(x, s)e^{sz}ds, \quad -\Re(P_1) < s_0 < 0. \tag{26}$$

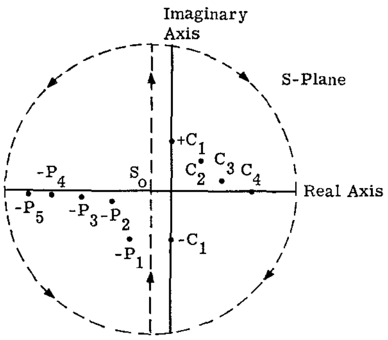

Fig. 2—Contours for evaluating (26).

The poles of $F(x, s)$ are sketched in Fig. 2. In order to evaluate (26) for $z<0$, the contour is closed at infinity in the right half plane, and the enclosed residues are evaluated. For $z>0$, the contour is closed in the left half plane. The residues in the right half plane yield terms containing $\cos \alpha_1 x e^{-C_1 z}$, $\cos \alpha_1 x e^{C_1 z}$, $\cos \alpha_2 x e^{C_2 z}$, $+ \cdots$ corresponding to the incident and reflected modes for $z<0$. The residues at $s=-P_n$ correspond to the transmitted modes. The entire field can then be written

$$E_{yI} = \cos \alpha_1 x e^{-C_1 z}, \text{ incident wave,} \tag{27}$$

$$E_{yR} = \sum_{n=1}^{\infty} R_n \cos \alpha_n x e^{C_n z}, \text{ reflected waves,} \tag{28}$$

$$E_{yT} = \sum_{n=1}^{\infty} T_n \cos \beta_n x e^{-P_n z}, \text{ transmitted waves.} \tag{29}$$

The coefficients R_1 and T_1 of the dominant modes are found from the ratios of the residues at $s=C_1$, and $s=-P_1$ to the residue at $s=-C_1$:

$$R_1 = \frac{C_1 - P_1}{P_1 + C_1}\prod_{n=2}^{\infty}\frac{(P_n - C_1)(C_n + C_1)}{(P_n + C_1)(C_n - C_1)} \tag{30}$$

$$T_1 = \frac{2C_1}{C_1 + P_1}\prod_{n=2}^{\infty}\frac{(P_n - C_1)(C_n + C_1)}{(P_n - P_1)(C_n + P_1)}. \tag{31}$$

The higher order terms T_n and R_n can be determined in a similar manner from the residues at $s=C_n$ and $s=-P_n$.

One word of caution is necessary here. In order to evaluate the inverse Laplace transform, we must have $-\Re(P_1)<s_0<0$; therefore, the propagation constant P_1 must have a real part which is greater than zero.

Although formulas (30) and (31) were evaluated for perfectly conducting walls, when $z<0$, they are also valid for any wall impedance Z_0 for $z<0$ provided that $\Re(C_1)<\Re(P_1)$. In this case the propagation constants C_n in the formulas will be those required to satisfy the boundary condition when $z<0$. Reflection and transmission coefficients for an incident TE_{r0} mode can also be obtained when $\Re(P_1)>\Re(C_r)$. Here (20) is multiplied by

$$(1 + s/C_r)\prod_{n=1}^{\infty}(1 + s/P_n)(1 - s/C_n)$$

rather than

$$(1 + s/C_1)\prod_{n=1}^{\infty}(1 + s/P_n)(1 - s/C_n).$$

Wave Incident from $z>0$

Since the derivation presented above is valid only when $\Re(P_1)>\Re(C_1)$, another solution is required when the incident wave lies on the lossy side of the junction. In this case one assumes that the transverse electric field is

$$E_y = F(x, z) + A \cos \beta_1 x e^{P_1 z}. \tag{32}$$

The term on the far right represents a wave incident from $z=+\infty$. The function $F(x, s)$ for this case is

$$F(x, s) = -\frac{\left[B + A\sum_{n=2}^{\infty}\frac{a_n}{s + C_n}\right]\cos Wx}{(1 + s/C_1)\prod_{n=1}^{\infty}(1 + s/P_n)(1 - s/C_n)} + \frac{A \cos \beta_1 a \cos Wx}{(s - P_1)\cos Wa}; \tag{33}$$

the a_n's are a series of constants. Again the inverse transform of $F(x, s)$ is used to find E_y. Here the constants B and A are selected so that the residue at $s=-C_1$ vanishes in order to eliminate the incident wave from $z<0$. The reflection and transmission coefficients of the dominant modes, R_1^* and T_1^*, are the ratios of the residues at $s=-P_1$ and $s=C_1$ to the constant A. The exact form of these coefficients is quite lengthy, but making the approximations $C_n \gg C_1$ and $C_n \gg P_1$, $n=2, 3, 4, \cdots$, these coefficients assume the simple forms

$$R_1^* \cong \frac{\alpha_1 \cos \beta_1 a T_1}{C_1 a(C_1 + P_1)} \tag{34}$$

$$T_1^* \cong \frac{\alpha_1 \cos \beta_1 a}{C_1 a}\left[\frac{R_1}{P_1 + C_1} + \frac{1}{P_1 - C_1}\right]. \tag{35}$$

TABLE I

REFLECTION AND TRANSMISSION COEFFICIENTS

$$\text{Wall Impedance} = \begin{cases} Z_1, & z > 0 \\ 0, & z < 0 \text{ (Walls have infinite conductivity)} \end{cases}$$

Wave incident from $z = -\infty$

$$R_1 = \frac{C_1 - P_1}{P_1 + C_1} \prod_{n=2}^{\infty} \frac{(P_n - C_1)(C_n + C_1)}{(P_n + C_1)(C_n - C_1)}$$

$$T_1 = \frac{2C_1K_1}{C_1 + P_1} \prod_{n=2}^{\infty} \frac{(P_n - C_1)(C_n + C_1)}{(P_n - P_1)(C_n + P_1)}$$

Wave incident from $z = +\infty$

$$R_1^* \cong \frac{K_2T_1}{C_1a(C_1 + P_1)}$$

$$T_1^* \cong \frac{K_2}{C_1a}\left[\frac{R_1}{P_1 + C_1} + \frac{1}{P_1 - C_1}\right]$$

Mode and Geometry	Eigenvalue Equations $z < 0$, $C_n = \sqrt{\alpha_n^2 - k_0^2}$	Eigenvalue Equations $z > 0$, $P_n = \sqrt{\beta_n^2 - k_0^2}$	K_1	K_2
TE_{10} in Rectangular Guide 2 lossy side walls Width $= 2a$	$\alpha_n = (n - 1/2)\pi/a$	$\cot \beta_n a = \frac{-j\beta_n Z_1}{\omega\mu}$	1	$\alpha_1 \cos \beta_1 a$
TE_{10} in Rectangular Guide 1 lossy side wall Width $= a$	$\alpha_n = \frac{n\pi}{a}$	$\tan \beta_n a = \frac{j\beta_n Z_1}{\omega\mu}$	$\frac{\alpha_1}{\beta_1}$	$\alpha_1 \sin \beta_1 a$
TE_{01} in Circular Guide Radius $= a$	$J_1(\alpha_n a) = 0$	$J_1(\beta_n a) = +\frac{j\beta_n Z_1 J_0(\beta_n a)}{\omega\mu}$	$\frac{\alpha_1}{\beta_1}$	$-\frac{\alpha_1 J_1(\beta_1 a)}{J_1'(\alpha_1 a)}$
TM_{01} in Circular Guide Radius $= a$	$J_0(\alpha_n a) = 0$	$J_0(\beta_n a) = \frac{j\omega\epsilon Z_1 J_1(\beta_n a)}{\beta_n}$	$\frac{\alpha_1}{\beta_1}$	$\frac{\beta_1 J_0(\beta_1 a)}{J_1(\alpha_1 a)}$

SIMILAR SITUATIONS

The derivations for R_1, T_1, R_1^*, and T_1^* for the TE_{10} mode in parallel-plane waveguide are also valid for a rectangular waveguide with two lossy side walls. Similar derivations hold for the TM_{01} and the TE_{01} modes in circular waveguide. Table I has the formulas for the reflection coefficients for these cases. The coefficients when the wave is incident from the lossless side are exact; the others are approximate.

Papadopoulos[3] presents the solution for the TE_{10} mode in rectangular waveguide in which a semi-infinite resistive strip is centrally placed. His results could also be entered in Table I except that he has used a slightly different approximation for R_1^* and T_1^* and that he appears to have some error in signs.

NUMERICAL EXAMPLE

To illustrate the application of these formulas to a practical situation, the reflection and transmission coefficients R_1 and T_1 were calculated for the TE_{10} mode in rectangular waveguide with one lossy side wall. These coefficients are plotted in Fig. 3 as a function of frequency for values of Z_1 equal to 20 and 200 Ω. Such a surface resistivity can be obtained by lining the side wall with a thin resistive card. The values of R_1 and T_1 were obtained by calculating the first seven terms of the infinite products for the 200 Ω resistive sheet, and assuming that the higher terms are all equal to unity. This is a good assumption since $P_n \gg C_1$, $C_n \gg C_1$, $P_n \gg P_1$, $C_n \gg P_1$, for large n. Only two terms were needed for the 20 Ω sheet. A guide width of 0.900″ was chosen for these calculations.

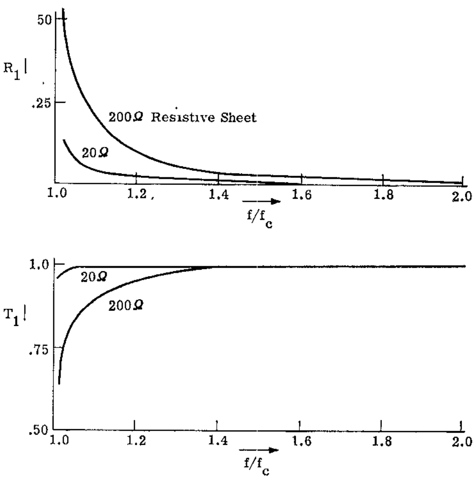

Fig. 3—$|R_1|$ and $|T_1|$ vs frequency for the TE_{10} mode in rectangular guide with a thin resistive sheet on one side (f_c=6560 Mc).

CONCLUSION

We have seen that the Wiener-Hopf method is useful for determining the transmission and reflection coefficients caused by a change in wall impedance of a waveguide. The form of the coefficients is similar for the TE_{10} mode in rectangular guide and for the TM_{01} and TE_{01} modes in circular guide. These coefficients contain infinite products; however, only a few of the terms need be computed in numerical calculations, since the higher order terms rapidly approach unity.

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